A Series of Lectures on
Approximate Dynamic Programming

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Our Aim

Discuss optimization by Dynamic Programming (DP)

and the use of approximations

Purpose: Computational tractability in a broad variety of practical contexts
The Scope of these Lectures

After an introduction to exact DP, we will focus on approximate DP for optimal control under stochastic uncertainty

- The subject is broad with rich variety of theory/math, algorithms, and applications
- Applications come from a vast array of areas: control/robotics/planning, operations research, economics, artificial intelligence, and beyond ...
- We will concentrate on control of discrete-time systems with a finite number of stages (a finite horizon), and the expected value criterion
- We will focus mostly on algorithms ... less on theory and modeling

We will not cover:

- Infinite horizon problems
- Imperfect state information and minimax/game problems
- Simulation-based methods: reinforcement learning, neuro-dynamic programming
- A series of video lectures on the latter can be found at the author’s web site

Reference: The lectures will follow Chapters 1 and 6 of the author’s book

## Lectures Plan

### Exact DP
- The basic problem formulation
- Some examples
- The DP algorithm for finite horizon problems with perfect state information
- Computational limitations; motivation for approximate DP

### Approximate DP - I
- Approximation in value space; limited lookahead
- Parametric cost approximation, including neural networks
- $Q$-factor approximation, model-free approximate DP
- Problem approximation

### Approximate DP - II
- Simulation-based on-line approximation; rollout and Monte Carlo tree search
- Applications in backgammon and AlphaGo
- Approximation in policy space
EXACT DYNAMIC PROGRAMMING
Outline

1. Basic Problem
2. Some Examples
3. The DP Algorithm
4. Approximation Ideas
Basic Problem Structure for DP

Discrete-time system

\[ x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, \ldots, N - 1 \]

- \( x_k \): State; summarizes past information that is relevant for future optimization at time \( k \)
- \( u_k \): Control; decision to be selected at time \( k \) from a given set \( U_k(x_k) \)
- \( w_k \): Disturbance; random parameter with distribution \( P(w_k \mid x_k, u_k) \)
- For deterministic problems there is no \( w_k \)

Cost function that is additive over time

\[ E \left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\} \]

Perfect state information

The control \( u_k \) is applied with (exact) knowledge of the state \( x_k \)
Feedback policies: Rules that specify the control to apply at each possible state $x_k$ that can occur.

Major distinction: We minimize over sequences of functions of state $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$, with $u_k = \mu_k(x_k) \in U_k(x_k)$ - not sequences of controls $\{u_0, u_1, \ldots, u_{N-1}\}$

Cost of a policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ starting at initial state $x_0$

$$J_\pi(x_0) = E \left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right\}$$

Optimal cost function:

$$J^*(x_0) = \min_{\pi} J_\pi(x_0)$$
Any optimization (deterministic, stochastic, minimax, etc) involving a sequence of decisions fits the framework

### A continuous-state example: Linear-quadratic optimal control

- **Linear discrete-time system:** \( x_{k+1} = Ax_k + Bu_k + w_k, \quad k = 0, \ldots, N - 1 \)
- \( x_k \in \mathbb{R}^n \): The state at time \( k \)
- \( u_k \in \mathbb{R}^m \): The control at time \( k \) (no constraints in the classical version)
- \( w_k \in \mathbb{R}^n \): The disturbance at time \( k \) (\( w_0, \ldots, w_{N-1} \) are independent random variables with given distribution)

### Quadratic Cost Function

\[
E \left\{ x_N' Q x_N + \sum_{k=0}^{N-1} (x_k' Q x_k + u_k' R u_k) \right\}
\]

where \( Q \) and \( R \) are positive definite symmetric matrices
Find optimal sequence of operations A, B, C, D (A must precede B and C must precede D)

**DP Problem Formulation**

- **States:** Partial schedules; Controls: Stage 0, 1, and 2 decisions
- **DP idea:** Break down the problem into smaller pieces (tail subproblems)
- **Start from the last decision and go backwards**
Solve the stage 2 subproblems (using the terminal costs)

At each state of stage 2, we record the optimal cost-to-go and the optimal decision
Scheduling Example Algorithm II

Solve the stage 1 subproblems (using the solution of stage 2 subproblems)

At each state of stage 1, we record the optimal cost-to-go and the optimal decision
Solve the stage 0 subproblem (using the solution of stage 1 subproblems)

- The stage 0 subproblem is the entire problem
- The optimal value of the stage 0 subproblem is the optimal cost $J^*(\text{initial state})$
- Construct the optimal sequence going forward
Let $\pi^* = \{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ be an optimal policy

Consider the “tail subproblem” whereby we are at $x_k$ at time $k$ and wish to minimize the “cost-to-go” from time $k$ to time $N$

$$E \left\{ g_N(x_N) + \sum_{m=k}^{N-1} g_m(x_m, \mu_m(x_m), w_m) \right\}$$

Consider the “tail” $\{\mu_k^*, \mu_{k+1}^*, \ldots, \mu_{N-1}^*\}$ of the optimal policy

THE TAIL OF AN OPTIMAL POLICY IS OPTIMAL FOR THE TAIL SUBPROBLEM

DP Algorithm

- Start with the last tail (stage $N - 1$) subproblems
- Solve the stage $k$ tail subproblems, using the optimal costs-to-go of the stage $(k + 1)$ tail subproblems
- The optimal value of the stage 0 subproblem is the optimal cost $J^*(\text{initial state})$
- In the process construct the optimal policy
Formal Statement of the DP Algorithm

Computes for all $k$ and states $x_k$: $J_k(x_k)$: opt. cost of tail problem that starts at $x_k$

Go backwards, $k = N - 1, \ldots, 0$, using

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} E_{w_k} \left\{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \right\}$$

Interpretation: To solve a tail problem that starts at state $x_k$

Minimize the $(k\text{-th-stage cost} + \text{Opt. cost of the tail problem that starts at state } x_{k+1})$

Notes:

- $J_0(x_0) = J^*(x_0)$: Cost generated at the last step, is equal to the optimal cost
- Let $\mu_k^*(x_k)$ minimize in the right side above for each $x_k$ and $k$. Then the policy $\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$ is optimal
- Proof by induction
The curse of dimensionality (too many values of $x_k$)

- In continuous-state problems:
  - Discretization needed
  - Exponential growth of the computation with the dimensions of the state and control spaces

- In naturally discrete/combinatorial problems: Quick explosion of the number of states as the search space increases

- Length of the horizon (what if it is infinite?)

The curse of modeling; we may not know exactly $f_k$ and $P(x_k | x_k, u_k)$

- It is often hard to construct an accurate math model of the problem
- Sometimes a simulator of the system is easier to construct than a model

The problem data may not be known well in advance

- A family of problems may be addressed. The data of the problem to be solved is given with little advance notice
- The problem data may change as the system is controlled – need for on-line replanning and fast solution
A MAJOR IDEA: Cost Approximation

IF we knew $J_{k+1}$, the computation of $J_k$ would be much simpler

- Replace $J_{k+1}$ by an approximation $\tilde{J}_{k+1}$
- Apply $\tilde{u}_k$ that attains the minimum in

$$\min_{u_k \in U_k(x_k)} E \left\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}(f_k(x_k, u_k, w_k)) \right\}$$

This is called one-step lookahead; an extension is multistep lookahead

A variety of approximation approaches (and combinations thereof):

- **Parametric cost-to-go approximation**: Use as $\tilde{J}_k$ a parametric function $\tilde{J}_k(x_k, r_k)$ (e.g., a neural network), whose parameter $r_k$ is “tuned” by some scheme
- **Problem approximation**: Use $\tilde{J}_k$ derived from a related but simpler problem
- **Rollout**: Use as $\tilde{J}_k$ the cost of some suboptimal policy, which is calculated either analytically or by simulation
ANOTHER MAJOR IDEA: Policy approximation

Parametrize the set of policies by a parameter vector \( r = (r_0, \ldots, r_{N-1}) \) (e.g., a neural network);

\[
\pi(r) = \{\tilde{\mu}_0(x_0, r_0), \ldots, \tilde{\mu}_{N-1}(x_{N-1}, r_{N-1})\}
\]

Minimize the cost \( J_{\pi(r)}(x_0) \) over \( r \)

A related possibility

- Compute a set of many state-control pairs \((x_k^s, u_k^s)\), \( s = 1, \ldots, q \), such that for each \( s \), \( u_k^s \) is a “good” control at state \( x_k^s \)
- Possibly use approximation in value space (or other “expert” scheme)
- Approximate in policy space by solving for each \( k \) the least squares problem

\[
\min_{r_k} \sum_{s=1}^{q} \| u_k^s - \tilde{\mu}_k(x_k^s, r_k) \|^2
\]

where \( \tilde{\mu}_k(x_k^s, r_k) \) is an “approximation architecture"

- A link between approximation in value and policy space
The connection of theory and algorithms (convergence, rate of convergence, complexity, etc) is solid for exact DP and most of optimization

By contrast, for approximate DP, the connection of theory and algorithms is fragile

Some approximate DP algorithms have been able to solve impressively difficult problems, yet we often do not fully understand why

There are success stories without theory

There is theory without success stories

The theory available is interesting but may involve some assumptions not always satisfied in practice

The challenge is how to bring to bear the right mix from a broad array of methods and theoretical ideas

Implementation is often an art; there are no guarantees of success

There is no safety in love, war, and approximate DP!