A Series of Lectures on Approximate Dynamic Programming

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Our Aim

Discuss optimization by Dynamic Programming (DP)

and the use of approximations

Purpose: Computational tractability in a broad variety of practical contexts

The Scope of these Lectures

After an intoduction to exact DP, we will focus on approximate DP for optimal control under stochastic uncertainty

- The subject is broad with rich variety of theory/math, algorithms, and applications
- Applications come from a vast array of areas: control/robotics/planning, operations research, economics, artificial intelligence, and beyond ...
- We will concentrate on control of discrete-time systems with a finite number of stages (a finite horizon), and the expected value criterion
- We will focus mostly on algorithms ... less on theory and modeling

We will not cover:

- Infinite horizon problems
- Imperfect state information and minimax/game problems
- Simulation-based methods: reinforcement learning, neuro-dynamic programming
- A series of video lectures on the latter can be found at the author's web site

Reference: The lectures will follow Chapters 1 and 6 of the author's book

"Dynamic Programming and Optimal Control," Vol. I, Athena Scientific, 2017

Lectures Plan

Exact DP

- The basic problem formulation
- Some examples
- The DP algorithm for finite horizon problems with perfect state information
- Computational limitations; motivation for approximate DP

Approximate DP - I

- Approximation in value space; limited lookahead
- Parametric cost approximation, including neural networks
- Q-factor approximation, model-free approximate DP
- Problem approximation

Approximate DP - II

- Simulation-based on-line approximation; rollout and Monte Carlo tree search
- Applications in backgammon and AlphaGo
- Approximation in policy space

EXACT DYNAMINC PROGRAMMING

Outline

Basic Problem Structure for DP

Discrete-time system

$$x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, 1, \ldots, N-1$$

- x_k: State; summarizes past information that is relevant for future optimization at time k
- u_k: Control; decision to be selected at time k from a given set $U_k(x_k)$
- w_k: Disturbance; random parameter with distribution $P(w_k \mid x_k, u_k)$
- For deterministic problems there is no w_k

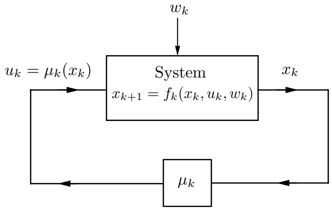
Cost function that is additive over time

$$E\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\}$$

Perfect state information

The control u_k is applied with (exact) knowledge of the state x_k

Optimization over Feedback Policies



- Feedback policies: Rules that specify the control to apply at each possible state x_k that can occur
- Major distinction: We minimize over sequences of functions of state $\pi = \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$, with $u_k = \mu_k(x_k) \in U_k(x_k)$ - not sequences of controls $\{u_0, u_1, \dots, u_{N-1}\}$

Cost of a policy $\pi = \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$ starting at initial state x_0

$$J_\pi(x_0) = E\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k\big(x_k, \mu_k(x_k), w_k\big) \right\}$$

Optimal cost function:

$$J^*(x_0) = \min_\pi J_\pi(x_0)$$

Scope of DP

Any optimization (deterministic, stochastic, minimax, etc) involving a sequence of decisions fits the framework

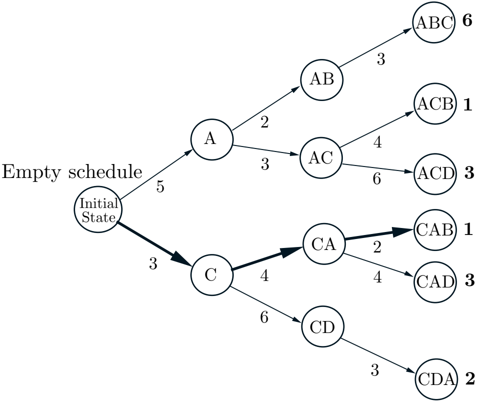
A continuous-state example: Linear-quadratic optimal control

- Linear discrete-time system: $x_{k+1} = Ax_k + Bu_k + w_k, \;\; k = 0, \ldots, N-1$
- $x_k \in \Re^n$: The state at time k
- $u_k \in \Re^m$: The control at time k (no constraints in the classical version)
- $w_k \in \Re^n$: The disturbance at time k ($w_0, \ldots, w_{N-1}$ are independent random variables with given distribution)

Quadratic Cost Function

$$E\left\{ x_N' Q x_N + \sum_{k=0}^{N-1} \left(x_k' Q x_k + u_k' R u_k \right) \right\}$$

where Q and R are positive definite symmetric matrices

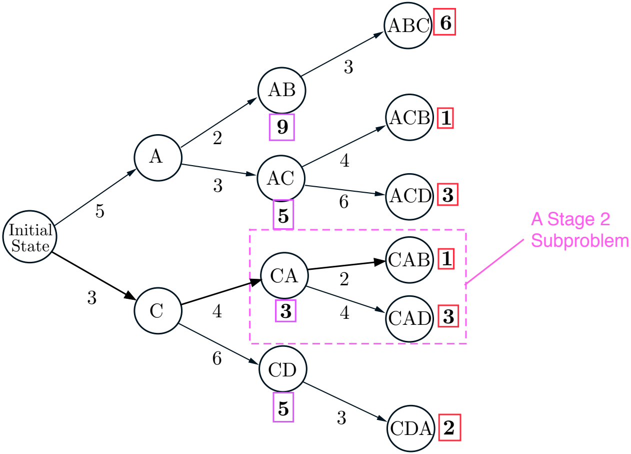


Find optimal sequence of operations A, B, C, D (A must precede B and C must precede D)

DP Problem Formulation

- States: Partial schedules; Controls: Stage 0, 1, and 2 decisions
- DP idea: Break down the problem into smaller pieces (tail subproblems)
- Start from the last decision and go backwards

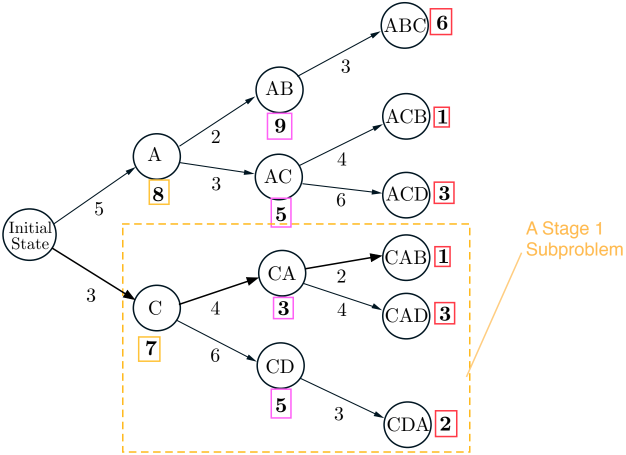
Scheduling Example Algorithm I



Solve the stage 2 subproblems (using the terminal costs)

At each state of stage 2, we record the optimal cost-to-go and the optimal decision

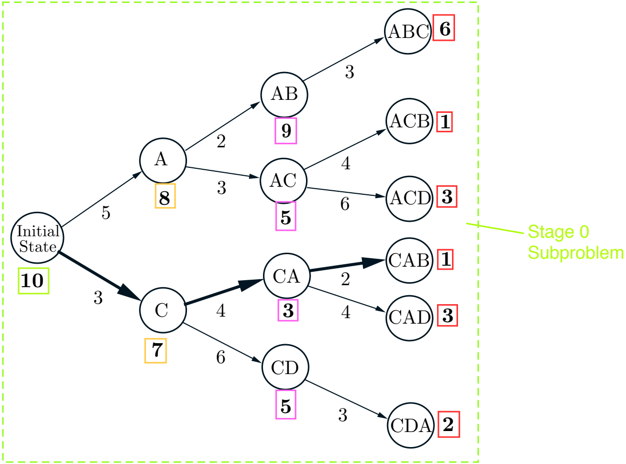
Scheduling Example Algorithm II



Solve the stage 1 subproblems (using the solution of stage 2 subproblems)

At each state of stage 1, we record the optimal cost-to-go and the optimal decision

Scheduling Example Algorithm III



Solve the stage 0 subproblem (using the solution of stage 1 subproblems)

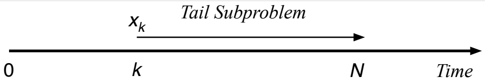
- The stage 0 subproblem is the entire problem
- The optimal value of the stage 0 subproblem is the optimal cost J^*(initial state)
- Construct the optimal sequence going forward

Principle of Optimality

- Let $\pi^* = \{\mu_0^*, \mu_1^*, \dots, \mu_{N-1}^*\}$ be an optimal policy
- Consider the "tail subproblem" whereby we are at x_k at time k and wish to minimize the "cost-to-go" from time k to time N

$$E\left\{ g_N(x_N) + \sum_{m=k}^{N-1} g_m\big(x_m, \mu_m(x_m), w_m\big) \right\}$$

Consider the "tail" $\{\mu_k^*, \mu_{k+1}^*, \dots, \mu_{N-1}^*\}$ of the optimal policy



THE TAIL OF AN OPTIMAL POLICY IS OPTIMAL FOR THE TAIL SUBPROBLEM

DP Algorithm

- Start with the last tail (stage $N-1$) subproblems
- Solve the stage k tail subproblems, using the optimal costs-to-go of the stage $(k+1)$ tail subproblems
- The optimal value of the stage 0 subproblem is the optimal cost J^*(initial state)
- In the process construct the optimal policy

Formal Statement of the DP Algorithm

Computes for all k and states x_k: $J_k(x_k)$: opt. cost of tail problem that starts at x_k

Go backwards, $k = N-1, \ldots, 0$, using

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big\}$$

Interpretation: To solve a tail problem that starts at state x_k

Minimize the (kth-stage cost + Opt. cost of the tail problem that starts at state x_{k+1})

Notes:

- $J_0(x_0) = J^*(x_0)$: Cost generated at the last step, is equal to the optimal cost
- Let $\mu_k^*(x_k)$ minimize in the right side above for each x_k and k. Then the policy $\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$ is optimal
- Proof by induction

Practical Difficulties of DP

The curse of dimensionality (too many values of x_k)

- In continuous-state problems:
  - Discretization needed
  - Exponential growth of the computation with the dimensions of the state and control spaces
- In naturally discrete/combinatorial problems: Quick explosion of the number of states as the search space increases
- Length of the horizon (what if it is infinite?)

The curse of modeling; we may not know exactly f_k and $P(x_k \mid x_k, u_k)$

- It is often hard to construct an accurate math model of the problem
- Sometimes a simulator of the system is easier to construct than a model

The problem data may not be known well in advance

- A family of problems may be addressed. The data of the problem to be solved is given with little advance notice
- The problem data may change as the system is controlled – need for on-line replanning and fast solution

Approximation in Value Space

A MAJOR IDEA: Cost Approximation

IF we knew J_{k+1}, the computation of J_k would be much simpler

- Replace J_{k+1} by an approximation $\tilde{J}_{k+1}$
- Apply $\bar{u}_k$ that attains the minimum in

$$\min_{u_k \in U_k(x_k)} E\Big\{g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}\big(f_k(x_k, u_k, w_k)\big)\Big\}$$

- This is called one-step lookahead; an extension is multistep lookahead

A variety of approximation approaches (and combinations thereoff):

- Parametric cost-to-go approximation: Use as $\tilde{J}_k$ a parametric function $\tilde{J}_k(x_k, r_k)$ (e.g., a neural network), whose parameter r_k is “tuned" by some scheme
- Problem approximation: Use $\tilde{J}_k$ derived from a related but simpler problem
- Rollout: Use as $\tilde{J}_k$ the cost of some suboptimal policy, which is calculated either analytically or by simulation

ANOTHER MAJOR IDEA: Policy approximation

Parametrize the set of policies by a parameter vector $r = (r_0, \ldots, r_{N-1})$ (e.g., a neural network);

$$\pi(r) = \{\tilde{\mu}_0(x_0, r_0), \ldots, \tilde{\mu}_{N-1}(x_{N-1}, r_{N-1})\}$$

Minimize the cost $J_{\pi(r)}(x_0)$ over r

A related possibility

- Compute a set of many state-control pairs (x_k^s, u_k^s), $s = 1, \ldots, q$, such that for each s, u_k^s is a “good" control at state x_k^s
- Possibly use approximation in value space (or other “expert" scheme)
- Approximate in policy space by solving for each k the least squares problem

$$\min_{r_k} \sum_{s=1}^{q} \left\| u_k^s - \tilde{\mu}_k(x_k^s, r_k) \right\|^2$$

where $\tilde{\mu}_k(x_k^s, r_k)$ is an “approximation architecture"

- A link between approximation in value and policy space

Perspective on Approximate DP

- The connection of theory and algorithms (convergence, rate of convergence, complexity, etc) is solid for exact DP and most of optimization
- By contrast, for approximate DP, the connection of theory and algorithms is fragile
- Some approximate DP algorithms have been able to solve impressively difficult problems, yet we often do not fully understand why
- There are success stories without theory
- There is theory without success stories
- The theory available is interesting but may involve some assumptions not always satisfied in practice
- The challenge is how to bring to bear the right mix from a broad array of methods and theoretical ideas
- Implementation is often an art; there are no guarantees of success
- There is no safety in love, war, and approximate DP!